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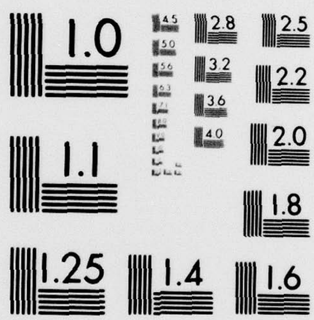

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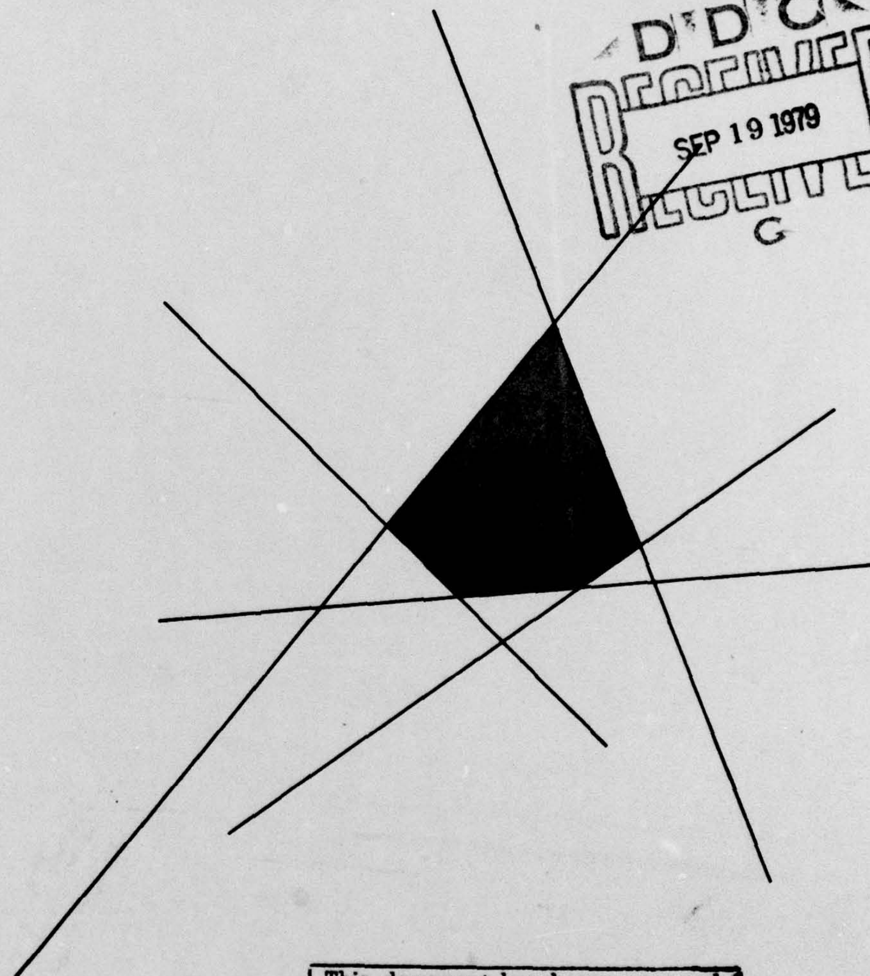
by
CYRUS DERMAN, GERALD J. LIEBERMAN and SHELDON M. ROSS



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(12)

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ABSTRACT

In the problem under consideration a decision maker has a total of M candidates to interview sequentially. The decision maker must either accept or reject the candidate being interviewed after he has been ranked with respect to his predecessors. Once rejected a candidate cannot be reconsidered; once a candidate is accepted no further interviews are carried out. The objective is to select the candidate in such a way as to maximize the probability of choosing the best of all M candidates (assuming every ordering of interviews is equally likely). In this paper we allow M to be a random variable and determine sufficient conditions on its distribution so as to obtain an optimal policy of simple form.

ON THE CANDIDATE PROBLEM WITH A RANDOM NUMBER OF CANDIDATES[1/]

by

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1. Introduction

In the problem under consideration a decision maker has a total of M candidates to interview sequentially. The decision maker must either accept or reject the candidate being interviewed after he has been ranked with respect to his predecessors. Once rejected a candidate cannot be reconsidered; once a candidate is accepted no further interviews are carried out. The objective is to select the candidate in such a way as to maximize the probability of choosing the best of all M candidates (assuming every ordering of interviews is equally likely).

In the classical version, $M = m$ is fixed and the optimal selection policy is to interview $r - 1$ candidates without choosing any; then select the first leading candidate interviewed thereafter. Here

$$r = r^*(m) = \min\left\{r \geq 1 \,\middle|\, \sum_{k=r}^{m-1} 1/k \leq 1\right\}.$$

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(See Chow, et.al. [1] p. 51 or Derman [3] p. 118.)

Rasmussen [5] and Rasmussen and Robbins [6] generalize the problem to the case where M is random with known probability function $P\{M = m\} = p_m$, $m = 1, \ldots, N$, $N < \infty$. The approach taken in [6] rests on the presumption (asserted in [5]) that the optimal solution policy possesses the above simple form and that one need only find the value of r that maximizes

$$\varphi(r) = \begin{cases} \sum_{m=1}^{N} p_m/m, & \text{if } r = 1 \\ (r-1) \sum_{m=r}^{N} p_m/m \sum_{k=r-1}^{m-1} 1/k, & \text{if } r \geq 2 \end{cases},$$

the a priori probability of selecting the best candidate under the rule which lets the first $r - 1$ go by and then selects the first leader that occurs thereafter.

The assertion of [5], however, is not true for every probability function $\{p_m\}$. In Section 2 a counter-example shows this. In Section 3 a sufficient condition is given for the optimal policy to have the simple form. The device for generating the condition is the "one-step look ahead" criterion given by Derman and Sacks [4] (also by Chow and Robbins [2]). It turns out that this criterion applies if and only if $\varphi(r)$ is unimodal. The condition given here for the criterion to apply improves on the one given in [6] for $\varphi(r)$ to be unimodal.

2. Counter-example

If $\{p_m\}$ were a two-point distribution with large probability mass at $M = 2$ (say) and small, but, positive, probability mass at $M = N$ (N large), a simple selection policy would have $r = 1$ or 2. Intuitively, however, it would seem that a better policy would be to select the second candidate if he was better than the first and if not to let a number of candidates go by before deciding to choose the next leading candidate. Formally, suppose

$$p_2 = 1 - \varepsilon, \; p_N = \varepsilon \; .$$

Then

$$\varphi(1) = \frac{1-\varepsilon}{2} + \varepsilon/N$$

$$\varphi(2) = \frac{1-\varepsilon}{2} + \frac{\varepsilon}{N} \sum_{k=1}^{N-1} 1/k$$

$$\varphi(r) < \varepsilon, \; 3 \leq r \leq N \; .$$

So for $N > 2$ and ε small

$$\varphi(2) = \max_r \varphi(r)$$

However, consider the policy that stops at $r = 2$, if possible, or, if $M = N$, at the first leading candidate after $i - 1$ candidates have been interviewed, where i shall be appropriately determined. Let φ denote the probability of selecting the best candidate under this policy. Then,

$$\varphi = P\{M = 2\} \quad P\{\text{Selecting Best Candidate} | M = 2\}$$

$$+ P\{M = N\} \; P\{\text{Selecting Best Candidate} | M = N\}$$

$$= \frac{1-\varepsilon}{2} + \varepsilon \left\{ 1/2 \cdot 2/N + 1/2 \cdot \frac{N-2}{N} \left(\frac{i-3}{N-2} \sum_{k=i-3}^{N-3} 1/k \right) \right\}$$

$$= \frac{1-\varepsilon}{2} + \varepsilon/N \left\{ 1 + \frac{i-3}{2} \sum_{k=i-3}^{N-3} 1/k \right\} .$$

Thus,

$$\varphi > \varphi(2)$$

if

$$1 + \frac{i-3}{2} \sum_{k=i-3}^{N-3} 1/k \geq \sum_{k=1}^{N-1} 1/k .$$

If i is any integer greater than 5 and N is sufficiently large, the latter inequality will hold.

3. One-step Look Ahead Policies

Consistent with our use of the expression in the foregoing, we say a leading candidate is being interviewed if he ranks above all previously interviewed candidates. A one-step look ahead policy is one that accepts a leading candidate if and only if the probability that he is the best exceeds the probability that one more leader will be interviewed and that he will be the best. Applying a theorem by Derman and Sacks [4] (also Chow and Robbins [2]), the one-step look ahead criterion yields an optimal selection policy if there is an $r - 1$ such that one-step look ahead accepts (rejects) a leader among the first j interviewed if $j > r - 1 (j \leq r - 1)$.

Let

$$p(m|i) = \frac{p_m}{\sum_{k \geq i} p_k}, \quad m \geq i,$$

denote the conditional probability that $M = m$ given $M \geq i$. Let $\mathcal{E}_j$ denote the event that the j^{th} candidate interviewed is a leader, $\mathcal{F}_j$ denote the event that the j^{th} candidate is the best. Then

$$p\{\mathscr{F}_j | M \geq j, \ \mathscr{E}_j\} = j \sum_{m=j}^{N} \frac{p(m|j)}{m} \equiv R(j) \qquad \text{(say)} .$$

Let $\mathscr{G}_j$ denote the event that the j^{th} candidate observed is a leader, one more leader will be observed and that he will be the best. Then

$$P\{\mathscr{G}_j | M \geq j, \ \mathscr{E}_j\} = \sum_{m=j+1}^{N} p(m|j) \sum_{k=j+1}^{m} \frac{j}{k(k-1)} \frac{k}{m}$$

$$= j \sum_{m=j+1}^{N} \frac{p(m|j)}{m} \sum_{k=j}^{m-1} 1/k$$

$$\equiv S(j) \qquad \text{(say)} .$$

Thus, the one-step look policy is optimal if for some $r - 1$

(1) $$R(j) - S(j) < 0, \ j \leq r - 1$$
$$\geq 0, \ j > r \quad .$$

Now, interpreting $\sum_{k=j}^{j-1} 1/k = 0$,

$$R(j) - S(j) = j\left\{\sum_{m=j}^{N} \frac{p(m|j)}{m} - \sum_{m=j+1}^{N} \frac{p(m|j)}{m} \sum_{k=j}^{m-1} 1/k\right\}$$

$$= \frac{j}{\sum_{m=j}^{N} p_m} \left\{ \sum_{m=j}^{N} \frac{p_m}{m} \left(1 - \sum_{k=j}^{m-1} 1/k\right)\right\}$$

$$= \frac{j}{\sum_{m=j}^{N} p_m} \psi(j)$$

where $\psi(j)$ is the expression within the last brackets. The sign of $R(j) - S(j)$ is the same as that of $\psi(j)$. Thus, from (1) follows

Proposition 1: A one-step look ahead policy is optimal if for some $r - 1$

(2) $$\psi(j) \leq 0, \ j = 1, \ldots, r - 1$$

$$\psi(j) > 0, \ j = r, \ldots, N .$$

The hypothesis of proposition 1 is equivalent to $\varphi(r)$ being unimodal since, as given in [6],

$$\varphi(j) - \varphi(j+1) = \psi(j), \ j = 1, \ldots, N - 1 .$$

Now let

$$H_j = p_j - \sum_{m=j+1}^{N} p_m/m, \ j = 1, \ldots, N - 1 .$$

In terms of H_j we state

Condition (*): $H_j \geq 0$ implies $H_i \geq 0$ for $i \geq j$.

We now have

<u>Theorem 1</u>: If condition (*) holds, then there is an r such that (2) holds (i.e., a one-stop look ahead policy is optimal).

<u>Proof</u>: In Rasmussen and Robbins [6] it is given that

$$\psi(j) - \psi(j+1) = H_j/j, \; j = 1, \ldots, N-1 .$$

Let i be any integer such that

$$\psi(i) \geq 0 .$$

If $H_i < 0$, then

$$\begin{aligned} \psi(i+1) &\geq \psi(i) \\ &\geq 0 . \end{aligned}$$

If $H_i \geq 0$, then by condition (*)

$$H_j \geq 0, \; j = i+1, \ldots, N-1,$$

and hence, $\psi(j)$ is non-increasing in j for all j, $i \leq j \leq N$. However,

$$\psi(N) \geq 0;$$

therefore, $\psi(j) \geq 0$ for all j, $i \leq j \leq N$. Thus, in both cases $\psi(i+1) \geq 0$, proving the theorem.

Remarks: Condition (*) is weaker than the condition $p_1 \leq p_2 \leq \cdots \leq p_N$ given in [6]. It is also satisfied by the geometric distribution over $1, \ldots, N$, a distribution for which $\varphi(r)$ is unimodal (as stated in [6]). It is not necessary to restrict N to being finite; under condition (*) Theorem 1 will hold with $N = \infty$ since, in this case, $\lim_{j\to\infty} \psi(j) = 0$.

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